TITLE: Supercavitating Nonlinear Flow Problems: Matched Asymptotics

Supercavitating Nonlinear Flow Problems: Matched Asymptotics

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Summary: The purpose of the lecture is to demonstrate the method of Matched Asymptotic Expansion (MAE) to be efficiently applicable to various supercavitating flow problems. Two model problems were chosen to prove the statement. The first one is a problem of the flow around a supercavitating shock free hydrofoil with spoiler mounted on its trailing edge and wedge-like fully wetted leading edge. A gallery of local nonlinear edge solution is proposed for the inner flow problems in the vicinity of the spoiler as well as in the vicinity of the sharp or rounded cavitating leading edge. The second problem is supercavitating wing of large aspect ratio beneath the free surface. In both cases analytical solutions to nonlinear inner problem were applied to accomplish the matching procedure and to significantly extend the range of the flow parameters where asymptotic approach gives reliable results. Solutions obtained in the framework of the MAE method as well as linear and nonlinear theories are illustrated in the text by numerical data.

1. Introduction

The Matched Asymptotic Expansions (MAE) method is an appropriate solution technique for a wide variety of fluid mechanics problems. As a rule, the classical MAE method involves the following steps while solving a lifting flow (including cavitating one) problem:

a) to determine what the small parameters of the problem are and, depending on the answer to the question, subdivide the flow domain into a so-called farfield and a nearfield, that is, regions far from and in the vicinity of a source of singular perturbations; formulation of problems in those regions – outer and inner problems correspondingly. The main goal of the subdividing procedure is to simplify the outer problem as compared to a general one (for instance, such a simplification is brought about by the linearization procedure for all the boundary conditions in outer region), and to take into account as much nonlinear effects as possible in the nearfield.

b) a 'rough' asymptotic analysis of both outer and inner descriptions from the viewpoint of a solution class to the problems arising in those subdomains. For instance, a presence of the spoiler on the trailing edge of planing or cavitating hydrofoil dictates a non-traditional class (namely, $\infty - \infty$ instead of customary $\infty - 0$) of linearized outer solution to the corresponding mixed boundary problem. In the case of a cavitating problem that means that the function of complex conjugate velocity in the farfield has to have square root singularity on the trailing edge of the hydrofoil and $-1/4$ on the leading one, see Rozhdestvensky & Fridman [17];

c) solution of the outer lifting flow problem, in the flow region far from the inner zone. Such an outer asymptotic expansion is often derived under the assumption of small perturbations brought by a hydrofoil into the inflow. The assumption loses its correctness in the vicinity of the inner sub-region and so does the outer expansion. Outer solution contains some elements of uncertainty caused by the influence of local problem, as some significant features were omitted from consideration.

Note that the MAE method does not always require the outer problem to be linear and it is not a general restriction on its applicability. Examples as such are abound but here we to some degree confine ourselves by such an assumption.

d) solution of the inner flow problem, i.e. construction of a so-called inner asymptotic expansion which is correct in the local region and loses its correctness in the farfield. It should be pointed out that inner problem is usually considered in stretched local coordinates, the stretching factor being connected with a chosen small parameter. It is by taking advantage of the stretching procedure that the inner problem is also simplified, for instance, reducing of three–dimensional problem to two dimensions in the nearfield, neglecting the gravity, *etc.* This allows to apply a nonlinear approach to adequately describe the most important part of the flow domain. Because of the influence of the outer region, inner solution contains some unknown parameters, as well;

e) matching procedure for outer and inner expansions which is applied to 'blend' them and to take into account their interaction. This stage allows us to close the solution to the whole problem because all the unknown

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parameters in the far- and nearfield are determined. Using the results obtained, a composite uniformly valid asymptotic solution to the problem under consideration is reached. Such a composite solution is shown to be correct in the whole flow domain and enables one to remove some intrinsic drawbacks of the purely linear description used in the outer region.

The advantages of such an asymptotic method of treating the lifting flow problems are easily seen and thoroughly discussed by many authors, see Van Dyke [24], Cole [2], Ogilvie [14], Rozhdestvensky [16], *etc.* In this connection a 'mathematical constructor' approach should be mentioned, advocated in Fridman [3] and Fridman & Rozhdestvensky [4, 17]. It is by applying this approach that one can pick up and then to combine the necessary outer and inner asymptotic expansions (just like standard 'mathematical parts' or the details of the whole construction) to obtain a desired everywhere correct solution.

It is a well known fact that the MAE method, being applied to the lifting flow problems, enables one to scrutinize those flow regions, where nonlinearities of the problem under consideration are concentrated and most pronounced, while outer flow domain, far from such zones of nonlinearity, can be described more or less sufficiently in the frameworks of an appropriate linear theory. In this case the outer description is certainly to retain some aftermaths of the linearization of the boundary conditions which, though, can be efficiently overcome by MAE technique.

2. Supercavitating shock free hydrofoil with spoiler

2.1 Problem formulation

Consider a cavitating problem for the hydrofoil with the spoiler, at arbitrary cavitation number, see figure 1, the influence of gravity being neglected. The Efros cavity closure model with re-entrant jet is adopted. The origin of the Cartesian coordinate system is taken at the plate's leading edge, x–axis being directed downstream and y upwards. There is an incident stream with speed V_∞ coming from the left. The region occupied by the fluid is bounded by the solid straight boundaries $[AC]$, $[OB]$ and arbitrarily arched portion (CO) given in the form $y = f(x)$ and the cavity surfaces (AE) and (BE), the intervals $[AC]$ and $[OB]$ being of the length $|AC| = l_{\rm w}$ and $|OB| = \varepsilon$. The hydrofoil chord $|OC|$ is chosen to be l. The incidence angle is α, the inclination angle of the spoiler with respect to the hydrofoil chord is β and the angle made by $[AC]$ and (CO) is γ.

It is of importance that the dividing streamline would meet the cavitating plate at the vertex of the wetted surface $(ACOB)$, namely at point C, to provide the shock–free cavitating mode. In this case the length of the upper wetted portion of the leading edge $l_{\rm w}$ is to be treated as an unknown parameter of the problem, the angle γ made by $[AC]$ and (CO) being given. Since the cavitation number $\sigma \geq 0$, the velocity absolute value on the free surfaces is $v_0 = v_\infty\sqrt{1+\sigma}$. The point at infinity on the re-entrant jet is denoted by E and is quite distinct from D, the point at infinity in the main stream. An additional stagnation point F appears in the flow pattern because of the re-entrant jet influence. That is why a double–sheeted Riemann surface is introduced, one sheet carrying the main flow pattern and the second including the jet. The direction of the jet at point E is determined by the angle μ, see figure 1. All the flow parameters are rendered nondimensional by using chord length l and velocity v_∞.

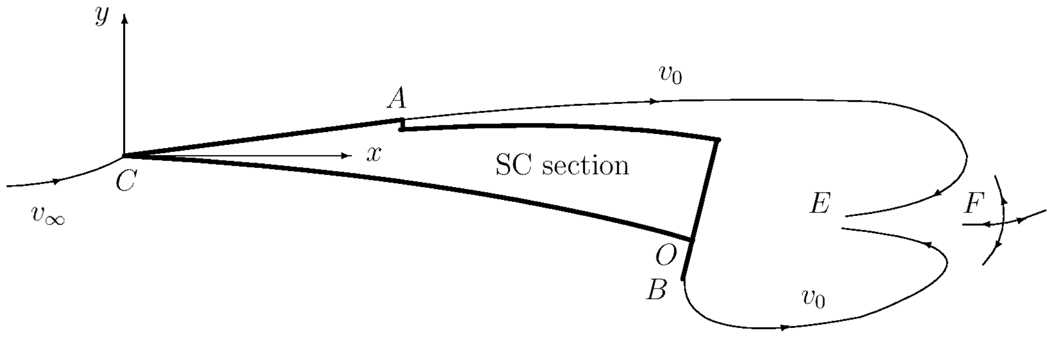


Figure 1: Shock free supercavitating hydrofoil with spoiler.

As is customary, the problem is considered to be solved when the velocity potential function $\varphi(x, y)$ is found. The harmonic function $\varphi(x, y)$, being the real part of an analytical function of complex potential $w = \varphi + i\psi$, has to satisfy the boundary kinematic (flow tangency) condition on the wetted length, the dynamic (pressure/velocity constancy) condition on the free surface and the condition at infinity.

Under assumption that the hydrofoil brings small perturbations into the inflow, subdivide the flow domain into a farfield (at distances of the order of O(1)) and nearfields (in the vicinity of the edges) we apply the MAE method.

2.2. Outer linear solution

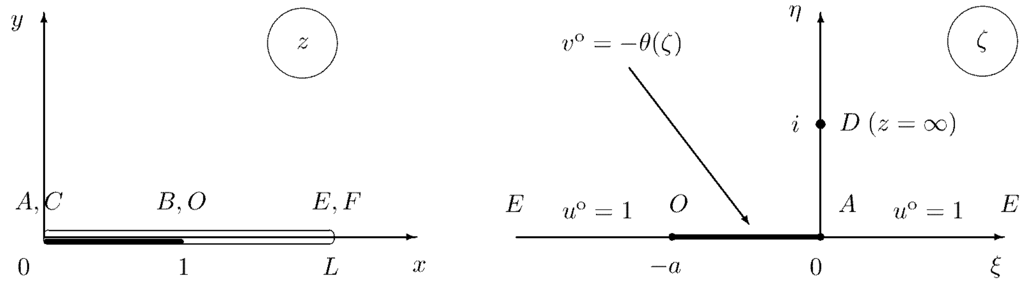
Assume that

$$\theta(x) = f'(x) = \mathrm{o}(1)\ , \quad \alpha = \mathrm{o}(1)\ , \quad \bar{\varepsilon} = \frac{\varepsilon}{l} = \mathrm{o}(1)\ , \quad \frac{l_{\mathrm{w}}}{l} = \mathrm{o}(\alpha)\ , \quad \sigma = \mathrm{o}(1)\ ,$$

that is, the hydrofoil and the cavity brings small perturbations into the inflow and therefore the linearization procedure has to be accomplished for the nonlinear boundary (dynamic and kinematic) conditions both on the free surfaces and on the wetted portion of the foil. Under those circumstances we can neglect the second order terms in all the equations and can formulate the linearized flow problem for supercavitating contour with spoiler [22, 21, 9, 10, 17]. All the wetted surface of the hydrofoil and the cavity appears as a slit of length L in the linearized plane $z = x + \mathrm{i}y$, where L actually is a linearized cavity extent. Boundary conditions for real and imaginary parts of the conjugate velocity χ^{o} are projected on the upper and lower boundaries of the slit, see figure 2, where

$$\chi^{\mathrm{o}}(z) = u^{\mathrm{o}} - i\, v^{\mathrm{o}} = \frac{\mathrm{d}w}{v_0\, \mathrm{d}z}\ ,$$

and $w(z) = \varphi + i\psi$ – function of the complex potential.



$$z = L\,\frac{\zeta^2}{1+\zeta^2}\ ; \quad \zeta = \sqrt{\frac{z}{L-z}}\ ; \quad a = \frac{1}{\sqrt{L-1}}$$

Figure 2: Physical and auxiliary planes.

The physical z-plane has to be mapped into the auxiliary upper semi-plane ζ where the mixed boundary value problem is formulated in $\infty - \infty$ class due to the presence of the spoiler located at the trailing edge of the foil. The class $\infty - \infty$ means that the function $\chi^{\mathrm{o}}(z)$ has to have square–root singularities for $z = 0$ and $z = 1$. Such a behaviour of the outer solution dictates by the asymptotic analysis of the flow problems arising in the vicinity of the leading and trailing edge with the spoiler done in the section that follows. The analysis also reveals that $l_{\mathrm{w}}/l \ll \alpha$ and therefore the kinematic boundary condition on the upper wetted portion $[AC]$ of the leading edge can be neglected correctly in the framework of the linear theory for all the wetted surface of the hydrofoil appears as a segment of unit length on the lower boundary of the slit in the physical z-plane (points A and C coincide and so do points B and O, and E and F).
Note that in the case of $l_{\mathrm{w}}/l = \mathrm{O}(1)$, the kinematic boundary condition on the upper wetted portion $[AC]$ has to be satisfied. To ensure shock free regime of the flow at the leading edge, the solution to the problem should behave at point C like $\sqrt{z}$ as $z \to 0$. Such a solution exists for unique value of the parameter l_{w}/l which becomes an unknown of the linear problem.
Back to the problem as $l_{\mathrm{w}}/l \ll \alpha$, apply Keldysh–Sedov formula for the upper semi-plane in $\infty - \infty$ class:

$$\chi^{\mathrm{o}}(\zeta) = 1 + \frac{i\,\mathcal{B}}{\sqrt{\zeta(\zeta+a)}} + \mathrm{i}\sqrt{\frac{\zeta+a}{\zeta}}\left(\frac{1}{\pi}\int\limits_0^a \sqrt{\frac{t}{a-t}}\,\frac{\theta(t)\,\mathrm{d}t}{\zeta+t} + \mathcal{A}\,\zeta + \mathcal{C}\right), \tag{1}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $a = 1/\sqrt{L-1}$ are unknown parameters to be determined from the cavity closure condition and condition at infinity

$$\mathrm{Im} \oint\limits_{\zeta=i} \chi^{\mathrm{o}}(\zeta) \frac{\mathrm{d}z}{\mathrm{d}\zeta} \, d\zeta = 0 \, ; \quad \chi^{\mathrm{o}}(i) = 1 - \frac{\sigma}{2} \, ,$$

along with condition provided by the matching procedure in the vicinity of the spoiler. It is to be underlined that the solution provides a square-root singularity for χ^{o} at cavity trailing edge $z = L$, $\zeta \to \infty$. It corresponds to the linearized closed cavity closure models: Riabouchinsky, Efros–Kreisel–Gilbarg, Tulin–Terentev (single spiral vortex).

Note that the simple non-quadrature approach can be proposed [21] for a wide range of the functions $y = f(x)$ characterizing the lower surface distribution of the hydrofoil.

It is obvious that the new function

$$\Omega(z) = \chi^{\mathrm{o}}(z) - 1 + \mathrm{i}\,\theta(z)$$

where $\theta(x) = f'(x)$ is a tangential angle to the foil at point x, has pure real values on the wetted portion (as $x \in [0; 1]$, $y = 0^{+}$) and pure imaginary values on the cavity surfaces, see figure 2. Let us assume that the wetted portion of the cavitating hydrofoil is a polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

and therefore

$$\theta(x) = f'(x) = \sum_{i=1}^{n} i a_i x^{i-1} \, .$$

Then function

$$\Omega(\zeta) = \chi^{\mathrm{o}}(\zeta) - 1 + \mathrm{i} \sum_{i=1}^{n} i a_i z^{i-1}(\zeta)$$

where $z(\zeta) = L\,\zeta^2/(1+\zeta^2)$, has to satisfy homogeneous boundary conditions on the upper semi-plane ζ in $\infty - \infty$ class. It should be pointed out that the multiplicity of a pole at the infinity $z \to \infty$ for the function $\Omega(z)$ is equal to $(n-1)$, i.e. to the degree of the polynomial $f'(x)$. That is why the solution can be derived without an integration procedure and is of the form

$$\chi^{\mathrm{o}}(\zeta) = 1 + \frac{\mathrm{i}\,\mathcal{B}}{\sqrt{\zeta(\zeta+a)}} - \mathrm{i}\theta(z(\zeta)) + \mathrm{i}\sqrt{\frac{\zeta+a}{\zeta}} \left\{ \mathcal{A}\zeta + \mathcal{C} + \sum_{k=1}^{n-1} \left(\frac{\mathcal{D}_k + i\mathcal{E}_k}{(\zeta-i)^k} + \frac{\mathcal{D}_k - i\mathcal{E}_k}{(\zeta+i)^k} \right) \right\} . \qquad (2)$$

Condition at infinity $z = \infty$ and at its image $\zeta = \mathrm{i}$ for function $\chi^{\mathrm{o}}(\zeta)$

$$\lim_{\zeta \to \mathrm{i}} \big(\chi^{\mathrm{o}}(\zeta) - 1 \big) = -\frac{\sigma}{2} \, ;$$

$$\lim_{\zeta \to \mathrm{i}} \Big\{ \big(\chi^{\mathrm{o}}(\zeta) - 1 \big) \, (\zeta - \mathrm{i})^k \Big\} = 0 \quad \text{for } k = 1, \ldots, n-1 \, ,$$

linearized cavity closure condition re-written in the form

$$\mathrm{Im} \, \frac{\mathrm{d}\chi^{\mathrm{o}}}{\mathrm{d}\zeta} \, (\mathrm{i}) = 0$$

along with the matching condition allow unknown parameters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}_k$, $\mathcal{E}_k$ and cavity extent L in (2) to be determined. So, the number $2n+2$ of unknowns coincides with that of conditions.

Once function $\chi^{\mathrm{o}}(\zeta)$ is found (i.e. the unknowns of the outer problem are derived), the cavity volume is determined

$$z^{+}_{\mathrm{cav}}(x) = \int\limits_0^x \chi^{\mathrm{o}}\Big(\sqrt{\frac{t}{L-t}} \Big) \mathrm{d}t \, , \qquad z^{-}_{\mathrm{cav}}(x) = \int\limits_1^x \chi^{\mathrm{o}}\Big(-\sqrt{\frac{t}{L-t}} \Big) \mathrm{d}t + 1 - \mathrm{i}\, f(1) \, , \qquad (3)$$

where superscripts $+$ and $-$ denote the upper and lower boundary of the cavity correspondingly.

All the hydrodynamic coefficients are also connected to $\chi^{\mathrm{o}}(z)$:

$$C_p(x) = -2\,\mathrm{Re}\left\{\chi^{\mathrm{o}}\Big(-\sqrt{\frac{x}{L-x}}\Big) - 1\right\} , \tag{4}$$

$$C_L = \int_0^1 C_p(x)\,\mathrm{d}x = 2\,\mathrm{Re} \oint_{\mathrm{i}} (\chi^{\mathrm{o}}(\zeta) - 1)\,\frac{\mathrm{d}z}{\mathrm{d}\zeta}\,\mathrm{d}\zeta , \tag{5}$$

$$C_D = \int_0^1 C_p(x)\,\theta(x)\,\mathrm{d}x . \tag{6}$$

Both classical and nonquadrature solutions to the supercavitating problem remain valid for the open cavity closure scheme (linearized analog of the Wu–Fabula model) as well. Such a model specifies the velocity field be continuous at the trailing edge of the cavity what means that the cavity and the trailing wake conjugates smoothly. From the mathematical point of view it corresponds to non-singular behaviour of $\chi^{\mathrm{o}}(z)$ at point $z = L$ which implies the condition $\mathcal{A} = 0$ to be satisfied. Then the number of unknowns becomes $2n + 1$ and the cavity closure condition $\mathrm{Im}\,\dfrac{\mathrm{d}\chi^{\mathrm{o}}}{\mathrm{d}\zeta}(\mathrm{i}) = 0$ should be neglected.

2.3. Inner nonlinear descriptions

2.3.1. A simplest spoiler problem. It was already mentioned above that outer asymptotic expansion loses its correctness in the vicinity of the spoiler as $z \to 1$ and inner description dictates the solution class of the outer one, namely $\infty - \infty$. The picture of the flow shown in figure 3 corresponds to stretching of local coordinates in this region by a factor $1/\bar{\varepsilon}$: $X = (x-1)/\bar{\varepsilon}$, $Y = y/\bar{\varepsilon}$, $\bar{\varepsilon} \to 0$. The region occupied by the fluid is bounded by the solid boundaries $[OB)$ and $[OA]$ and the free surface (AB), the interval $[OA]$ being of the unit length $|OA| = 1$. The absolute value of velocity V_∞^i at 'local' infinity and on the free surface (AB) is an unknown parameter to be determined through the matching procedure with outer problem. This is the simplest flow problem for a straight spoiler.

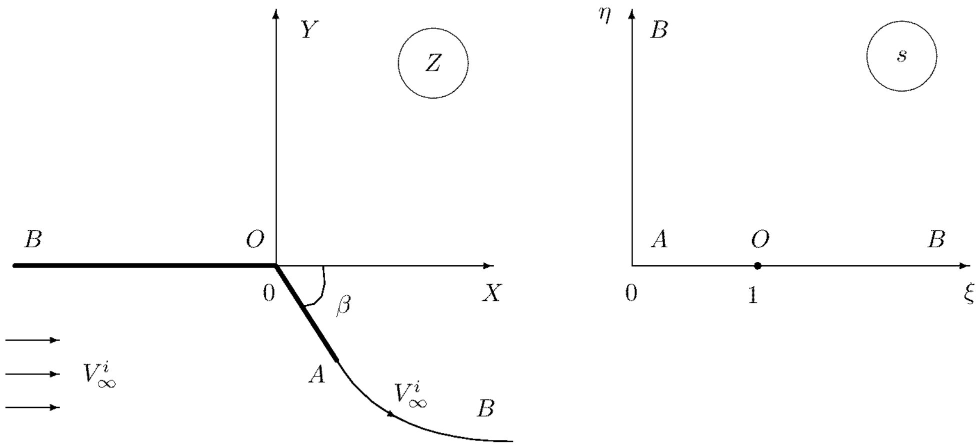


Figure 3: Flow pattern and auxiliary ζ-plane for the inner spoiler problem.

With the correspondence between physical and auxiliary planes (the first quadrant of the $\zeta = \xi + \mathrm{i}\eta$ plane is chosen as an image of the region occupied by the fluid), one can easily obtain the analytical inner nonlinear solution (following to Chaplygin method of singular points) in the form:

$$\chi^i(s) = \frac{\mathrm{d}w}{V_\infty^i \mathrm{d}Z} = \left(\frac{\zeta - 1}{\zeta + 1}\right)^{\beta/\pi} ; \qquad \frac{\mathrm{d}w}{\mathrm{d}s} = N\,\zeta ; \tag{7}$$

wherefrom

$$Z(s) = \frac{N}{V^i_\infty} \int_1^\zeta s \left(\frac{\zeta+1}{\zeta-1}\right)^{\beta/\pi} \mathrm{d}\zeta\ , \tag{8}$$

where N – unknown parameter to be determined from the condition $Z_A = Z(0) = \mathrm{e}^{-\mathrm{i}\beta}$ which yields

$$-\frac{N}{V^i_\infty} \int_0^1 \xi \left(\frac{1+\xi}{1-\xi}\right)^{\beta/\pi} \mathrm{d}\xi = -\frac{N_1}{U_1} R = 1\ . \tag{9}$$

The pressure distribution coefficient ($0 \le \xi < \infty$) is

$$C^i_p = 1 - \left|\chi^i(\zeta)\right|^2 = 1 - \left|\frac{\xi-1}{\xi+1}\right|^{2\beta/\pi}\ , \tag{10}$$

where the value of ξ is obtained from the relationships between physical and auxiliary coordinates on the interval $[OB]$, see figure 3:

$$X(\eta) = \frac{N}{V^i_\infty} \int\limits_1^\xi \xi\Big(\frac{\xi+1}{\xi-1}\Big)^{\beta/\pi} \mathrm{d}\xi\ , \qquad \xi \ge 1 \tag{11}$$

and on the interval $[OA]$:

$$|Z(\xi)| = -\frac{N}{V^i_\infty} \int\limits_\xi^1 \xi\Big(\frac{1+\xi}{1-\xi}\Big)^{\beta/\pi} \mathrm{d}\xi\ , \quad 0 \le \xi \le 1 \tag{12}$$

The value of V^i_∞ is the only unknown parameter left in this problem to be determined through the matching procedure.
It follows from (11) that $X \sim N\xi^2/(2V^i_\infty)$ as $X \to -\infty$ and $\xi \to +\infty$ and therefore velocity behaves as $X \to -\infty$

$$\frac{\mathrm{d}w}{\mathrm{d}Z} \sim V^i_\infty \left(1 - \frac{2\beta}{\pi}\frac{1}{\xi}\right) \sim V^i_\infty \left(1 - \frac{\sqrt{2}\,\beta}{\pi}\sqrt{\frac{-N}{V^i_\infty}}\,\frac{1}{\sqrt{-X}}\right). \tag{13}$$

If one assumes that $V^i_\infty = 1$, then coefficient $\mathcal{K}$ is

$$\mathcal{K} = \frac{\sqrt{2}\,\beta}{\pi}\sqrt{\frac{-N}{V^i_\infty}} = \frac{\sqrt{2}\,\beta}{\pi}\frac{1}{\sqrt{R}}\ ,$$

see figure 4 for $\mathcal{K}(\beta)$ curve, which attains its maximum at $\beta \approx 1.734$ (or 99.34°).

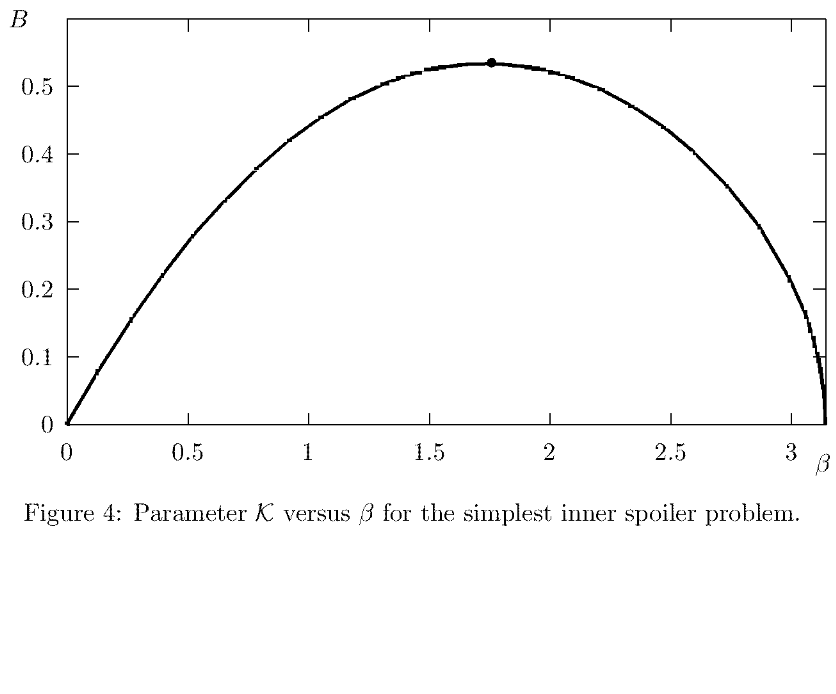


Figure 4: Parameter $\mathcal{K}$ versus β for the simplest inner spoiler problem.

2.3.2. Cavitating shock free leading edge Introduce a local stretched coordinate system in the vicinity of the wedge-like leading edge $Z = X + \mathrm{i}Y = z/l_{rmw}$, where $l_{rmw} \ll \alpha$ is the length of the upper wetted portion of the leading edge. As stagnation point coincides with the vertex of the wedge, value of l_{rmw} is unknown, but can be used as a stretching factor.
The flow pattern and corresponding auxiliary ζ-plane are depicted in figure 5. The flow region is bounded by semi-infinite line (OB), segment $[OC]$ and free surface (CD). Note that the interval $[OC]$ is of unit length because of the choice of the stretching factor. Like for the inner spoiler problem, the absolute value of velocity V_∞^i at 'local' infinity and on the free surface (CB) is an unknown parameter to be determined through the matching procedure with outer problem.

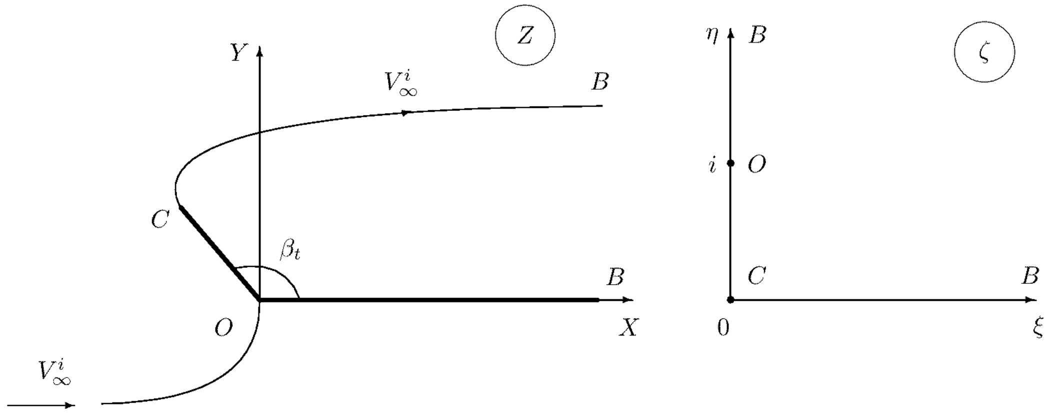


Figure 5: Flow pattern and auxiliary ζ-plane for the inner leading edge problem.

With the correspondence between physical and auxiliary planes (the first quadrant of the $\zeta = \xi + \mathrm{i}\eta$ plane is chosen as an image of the region occupied by the fluid), one can easily obtain the analytical inner nonlinear solution (following to Chaplygin method of singular points) in the form:

$$\chi^i(\zeta) = \frac{\mathrm{d}w}{V_\infty^i \mathrm{d}Z} = \left(\frac{\zeta - \mathrm{i}}{\zeta + \mathrm{i}}\right)^{\gamma/\pi} ; \qquad \frac{\mathrm{d}w}{\mathrm{d}\zeta} = N\zeta(\zeta^2 + 1) ;$$
$$Z(\zeta) = \frac{N}{V_\infty^i} \int\limits_{\mathrm{i}}^{\zeta} \zeta\,(\zeta^2 + 1) \left(\frac{\zeta + \mathrm{i}}{\zeta - \mathrm{i}}\right)^{\gamma/\pi} \mathrm{d}\zeta \, . \tag{14}$$

The length of the segment $[OC]$ ($|OC| = 1$) is connected to the unknown parameter N through the relation

$$|CO| = \frac{N}{V_\infty^i} \int\limits_0^1 t\,(1 - t^2) \left(\frac{1 + t}{1 - t}\right)^{\gamma/\pi} \mathrm{d}t = \frac{N}{V_\infty^i}\, Q \, . \tag{15}$$

It is seen that $X \sim N\eta^4/(4V_\infty^i)$ as $X \to +\infty$ and $Y = 0^-$ ($\xi = 0$, $\eta \to +\infty$). Therefore the asymptotic structure of the inner solution at 'local' infinity as $X \to +\infty$ and $Y = 0^-$ is

$$\left|\frac{\mathrm{d}w}{\mathrm{d}Z}\right| \sim V_\infty^i \left(1 - \frac{2\gamma}{\pi}\frac{1}{\eta}\right) \sim V_\infty^i \left(1 - \frac{\sqrt{2}\gamma}{\pi} \sqrt[4]{\frac{N}{V_\infty^i}} \frac{1}{X^{1/4}}\right) . \tag{16}$$

Such a behaviour corresponds to a well-known singularity '$-1/4$' arising in the framework of the linear cavitation theory at the leading edge of the hydrofoil.
Note that in the case of $\gamma = \pi$ the inner problem reduces to a well-known problem of the flow in the vicinity of the sharp leading edge of the cavitating hydrofoil considered by Plotkin [15] on the base of a hodograph method,

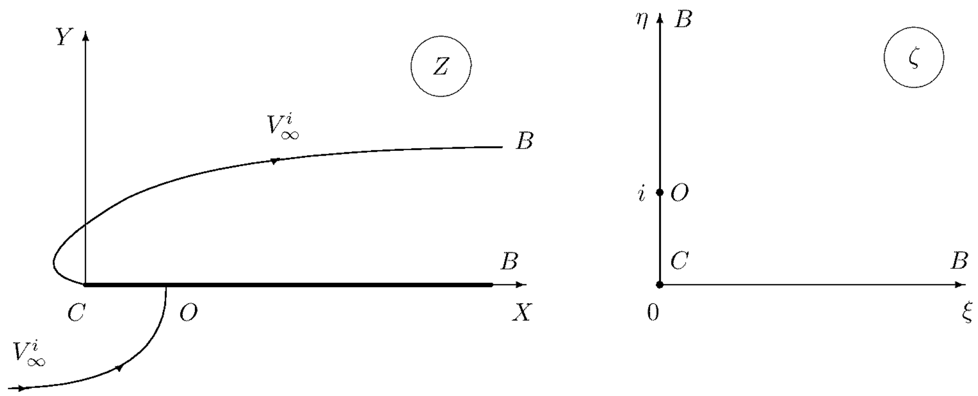


Figure 6: Flow pattern and auxiliary ζ-plane for the inner leading edge problem in the case of $\gamma = \pi$.

see figure 6. The solution presented below is appeared to be more efficient and convenient:

$$\chi^i(\zeta) = \frac{\mathrm{d}w}{V^i_\infty \mathrm{d}Z} = \frac{\zeta - \mathrm{i}}{\zeta + \mathrm{i}}; \qquad \frac{\mathrm{d}w}{\mathrm{d}\zeta} = N\zeta(\zeta^2 + 1)$$
$$Z(\zeta) = \frac{12}{17}\mathcal{L}\Big(\frac{1}{4}\zeta^4 + \frac{2}{3}\mathrm{i}\zeta^3 - \frac{1}{2}\zeta^2\Big); \tag{17}$$

where $\mathcal{L} = |OC|$ and, since $X \sim (3/17)\, L\eta^4$ as $X \to +\infty$ and $\eta \to +\infty$,

$$\left|\frac{\mathrm{d}w}{\mathrm{d}Z}\right| \sim V^i_\infty\Big(1 - \frac{2}{\eta}\Big) \sim V^i_\infty\Big(1 - \sqrt[4]{\frac{48}{17}\mathcal{L}}\,\frac{1}{X^{1/4}}\Big) \tag{18}$$

as $X \to +\infty$, $Y = 0^-$. The value of $\mathcal{L}$ and V^i_∞ are to be determined through the matching procedure.

2.3.3. Cavity closure region It is obvious that outer linear solution loses its correctness in the vicinity of the cavity trailing edge as $z \to L$, where L is the cavity extent, provided a linearized closed cavity closure scheme (like Riabouchinsky, Tulin–Terentev or Efros–Kreizel–Gilbarg models) is adopted. The outer expression for the conjugate velocity $\chi^{\mathrm{o}}(z)$ has a square-root singularity at $z = L$.
Since the latter linearized model is adopted in the outer region, a local nonlinear model is proposed of the flow in the close proximity of the re-entrant jet, see figure 7. The *upper half* of the flow pattern is shown in stretched and rotated local coordinates, the stretching factor being $1/\delta_e$, where δ_e is re-entrant jet width at infinity and rotating angle being γ_e, where angle γ_e describes a direction of re-entrant jet at infinity for Efros–Kreisel–Gilbarg cavity closure scheme. Since the flow pattern has axial symmetry, the streamline (AC) is substituted by the straight solid boundary (ABC). Then the region occupied by the fluid is bounded by free surface (AFC) and solid horizontal line (ABC), velocity vector at point D is directed downward. The dividing streamline meets the solid wall at stagnation point B. This streamline subdivides the flow into two parts: the first one is the main flow directed downstream with velocity V^i_∞ at local infinity as $X \to +\infty$ and the second one forms a re-entrant jet of unit width at infinity at point A as $X \to -\infty$.
With the correspondence between the physical $z = x + \mathrm{i}y$ plane and auxiliary quadrant $\zeta = \xi + \mathrm{i}\eta$ shown in figure 7, the Chaplygin method allows us to write down the following solution:

$$\chi^i(\zeta) = \frac{\mathrm{d}w}{V^i_\infty \mathrm{d}Z} = \frac{\zeta - 1}{\zeta + 1}; \qquad \frac{\mathrm{d}w}{\mathrm{d}\zeta} = N\frac{\zeta^2 - 1}{\zeta}; \tag{19}$$

$$Z(\zeta) = \int_1^\zeta \frac{\mathrm{d}Z}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}\zeta}\mathrm{d}\zeta = \frac{N}{V^i_\infty}\left(\frac{\zeta^2}{2} + 2\zeta + \log\zeta - 2.5\right), \tag{20}$$

where unknown parameter N is connected to the flow rate in the jet at point A through the relation

$$N = \frac{2V^i_\infty}{\pi}\,. \tag{21}$$

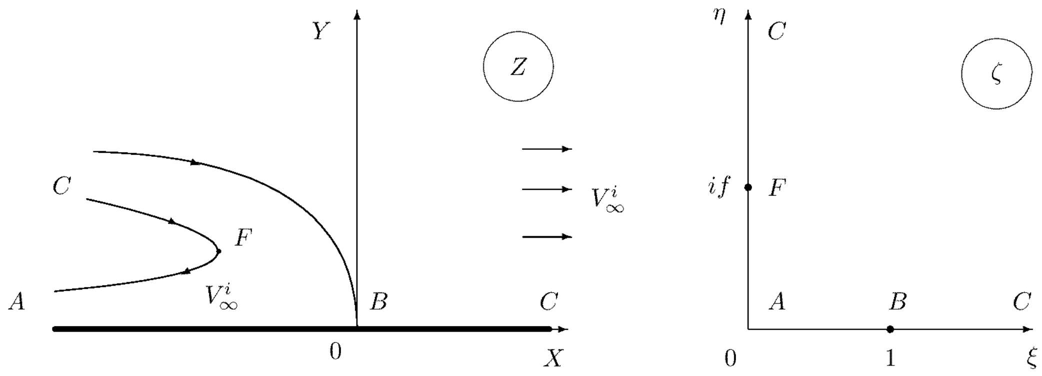


Figure 7: Flow pattern and auxiliary ζ-plane for the inner cavity closure problem.

It readily follows from condition $\chi^i(\mathrm{i}f) = i$ that $f = 1$ what gives the distance L^* between stagnation point B and normal projection of point F onto the horizontal line (ABC):

$$L^* = |\mathrm{Re}\ (Z(\mathrm{i}))| = \frac{6}{\pi}\,. \tag{22}$$

Both δ_e and V^i_∞ are to be determined through the matching procedure with outer linear solution.
It is seen that $X \sim \xi^2/\pi$ as $X \to +\infty$ and $\xi \to +\infty$, that is why

$$\frac{\mathrm{d}w}{\mathrm{d}Z} \sim V^i_\infty \left(1 - \frac{2}{\xi}\right) \sim V^i_\infty \left(1 - \frac{2}{\sqrt{\pi X}}\right), \tag{23}$$

which corresponds to the square-root singularity of the conjugate velocity for the outer linear expansion at the trailing edge of the cavity at $z = L$.

2.4. The matching procedure

It was mentioned above that outer solution loses its correctness in the vicinity of the hydrofoil and cavity edges ($z \to 0$, $z \to 1$ and $z \to L$) where function $\chi^o(z)$ has singularities of 1/2 (square root) and 1/4 type. More to the point, the outer solution is not completed, for the number of unknowns is greater then the number of conditions and coefficient $\mathcal{B}$ in (1) or (2) is an unknown parameter. The main goal of the matching procedure is to close the whole problem (to determine all the parameters in outer and inner descriptions) and to get rid of the singularities of the linear outer solution.
Generally speaking, the matching procedure is carried out into three steps: the first one allows us to match asymptotic descriptions in the outer region and in the trailing edge region (the velocity $U_1 = V^i_\infty$ in expression (7) and the coefficient $\mathcal{B}$ in the equation (1) or (2) are determined); the second one matches the outer description with known value of $\mathcal{B}$ and leading edge expansion to yield the values of l_w and $U_2 = V^i_\infty$ in (14) and the third one enables re-entrant jet width δ_e and velocity $U_3 = V^i_\infty$ in (19) to be derived. Notation U_1, U_2 and U_3 is introduced to distinguish velocity absolute value in the local regions.
First, it is obvious that

$$\xi + a \sim \frac{La^3}{2}(1 - x)\,,$$

as $x \to 1^-$, $y = 0^-$ and $\xi \to -a^+$, $\eta = 0^+$ (see figure 2) and therefore the limiting form of (1) and (2) near the trailing edge with spoiler is

$$\chi^{oi}_1(x) \sim 1 + \frac{\mathcal{B}}{\sqrt{a(a+\xi)}} \sim 1 + \frac{\sqrt{2}\,\mathcal{B}}{a^2\sqrt{L}}\,\frac{1}{\sqrt{1-x}}\,,$$

where $L = \cos^{-2}(\tau/2) > 1$ and $a = \cot(\tau/2)$.

Taking into account the fact that $X = (x-1)/\bar{\varepsilon}$, where $\bar{\varepsilon} = \varepsilon/l$ – is relative spoiler length this expression and formula (13) are compared to give

$$U_1 = V_0 = V_\infty\sqrt{1+\sigma}\ ;$$
$$\mathcal{B} = -\frac{\beta}{\pi}\cos\frac{\tau}{2}\sin^{-2}\frac{\tau}{2}\sqrt{\frac{\bar{\varepsilon}}{R}}\ , \tag{24}$$

where

$$R = \int\limits_0^1 \xi\Big(\frac{1+\xi}{1-\xi}\Big)^{\beta/\pi}\mathrm{d}\xi\,.$$

It is to be underlined that all the coefficients $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}_k$, $\mathcal{E}_k$ in (1) and (2) are of the same order of magnitude. That is why $\bar{\varepsilon} = \mathrm{O}(\alpha^2)$ as $\mathcal{A} = \mathrm{O}(\alpha)$.
Second, it is seen that

$$\xi \sim -\sqrt{\frac{x}{L}}\ ,$$

as $x \to 0^+$, $y = 0^-$ and $\xi \to 0^-$, therefore the limiting form of the classical solution (1) near the leading edge is

$$\chi_2^{\mathrm{o}i}(x) \sim 1 + \mathrm{i}\sqrt{\frac{a}{\xi}}\left(\frac{1}{\pi}\int\limits_0^a \frac{\theta(t)}{\sqrt{t(a-t)}}\mathrm{d}t + \frac{\mathcal{B}}{a} + \mathcal{C}\right) \sim 1 + \sqrt[4]{\frac{L}{a^2}}\left(\frac{a}{\pi}\int\limits_0^a \frac{\theta(t)}{\sqrt{t(a-t)}}\,dt + \mathcal{B} + a\mathcal{C}\right)\frac{1}{\sqrt[4]{x}}\ ,$$

and for nonquadrature solution (2) is

$$\chi_2^{oi}(x) \sim 1 + i\sqrt{\frac{a}{\xi}}\left\{\frac{\mathcal{B}}{a} + \mathcal{C} - 2\left(\mathcal{E}_1 + \mathcal{D}_2 - \mathcal{E}_3 - \mathcal{D}_4 + \mathcal{E}_5 + \mathcal{D}_6 - \ldots\right)\right\} \sim$$
$$\sim 1 + \sqrt[4]{\frac{L}{a^2}}\frac{1}{\sqrt[4]{x}}\left\{\mathcal{B} + a\mathcal{C} - 2a\left(\mathcal{E}_1 + \mathcal{D}_2 - \mathcal{E}_3 - \mathcal{D}_4 + \mathcal{E}_5 + \mathcal{D}_6 - \ldots\right)\right\}\,,$$

the value of the coefficient B being given by (24).
The matching procedure for these expansions and expression (16) considering the fact that $Z = z/(l_\mathrm{w}/l)$ results in

$$U_2 = V_0 = V_\infty\sqrt{1+\sigma}\ ;$$
$$\frac{l_\mathrm{w}}{l} = \frac{\pi^4 Q}{4\gamma^4}\frac{L}{a^2}\Big\{\mathcal{B} + a\mathcal{C} - 2a\left(\mathcal{E}_1 + \mathcal{D}_2 - \mathcal{E}_3 - \mathcal{D}_4 + \mathcal{E}_5 + \mathcal{D}_6 - \ldots\right)\Big\}^4 \tag{25}$$

where

$$Q = \int\limits_0^1 t\,(1-t^2)\left(\frac{1+t}{1-t}\right)^{\gamma/\pi}\mathrm{d}t\,.$$

Note that $\mathcal{D}_k = \mathcal{E}_k = 0$, $k = 1, 2, \ldots$ in the case of the supercavitating shock free flat plate. Expressions (25) show the influence of flow parameters including spoiler geometry onto the length of the upper wetted portion of the leading edge l_w for a given value of angle γ. It follows from the latter equation (25) that $l_\mathrm{w}/l = \mathrm{O}(\alpha^4)$ as $\mathcal{C}$, $\mathcal{D}_k$, $\mathcal{E}_k$ are of the order of $\mathrm{O}(\alpha)$ and $\mathcal{B} = \mathrm{O}(\sqrt{\bar{\varepsilon}}\,) = \mathrm{O}(\alpha)$.
Third, in the vicinity of the cavity trailing edge $x \to L^-$, $y = 0^+$ and $\xi \to +\infty$, $\eta = 0^+$ (see figure 2)

$$\xi \sim \sqrt{\frac{L}{L-x}}\ ,$$

wherefrom the limiting form of outer solution (both classical and nonquadrature) is

$$\chi_3^{\mathrm{o}i}(x) \sim 1 + \mathrm{i}\mathcal{A}\xi \sim 1 + \mathrm{i}\mathcal{A}\sqrt{\frac{L}{L-x}}\ .$$

Taking into account the fact that $X = (x-L)/(\delta_e/l)$, where δ_e/l is a half of the relative width of the re-entrant jet, this expression along with (23) are compared to yield

$$U_3 = V_0 = V_\infty\sqrt{1+\sigma}\ ;\qquad \frac{\delta_e}{l} = \frac{\pi}{4}\,L\,\mathcal{A}^2\,. \tag{26}$$

The expressions show relation between the width of the re-entrant jet and flow parameters (including spoiler length $\bar{\varepsilon}$ and inclination angle β). Note that $\delta_e/l = \mathrm{O}(\alpha^2)$ because of $\mathcal{A} = \mathrm{O}(\alpha)$.
The 'information stream' in the matching procedure is directed from the trailing edge of the hydrofoil to the leading one and to the trailing edge of the cavity as well. In fact, the spoiler geometry parameters β and $\bar{\varepsilon}$ dictate the value of the coefficient $\mathcal{B}$ which, in turn, defines the leading edge and re-entrant jet characteristics. The additive composite solution for the conjugate velocity can be constructed in a following manner:

$$\chi^{\mathrm{c}}(z) = \chi^{\mathrm{o}}(z) + \chi_1^{\mathrm{i}}(z) + \chi_2^{\mathrm{i}}(z) + \chi_3^{\mathrm{i}}(z) - \chi_1^{\mathrm{oi}}(z) - \chi_2^{\mathrm{oi}}(z) - \chi_3^{\mathrm{oi}}(z)\,, \tag{27}$$

where subscripts 1, 2, 3 correspond to local flow problems in the vicinity of the spoiler, leading edge and cavity trailing edge. Substituting composite solution χ^{c} instead of outer χ^{o} into formulae (3)–(6) enables one to derive hydrodynamic coefficients which are everywhere valid and to calculate correct flow pattern.

2.5. Exact solution to the nonlinear flat plate problem and asymptotic analysis

Consider a nonlinear problem of the flow around a supercavitating shock free flat plate. It is useful as a verification of the asymptotic results obtained in the previous sections. The problem under consideration is that of the theory of jets in an ideal fluid and has to be treated by corresponding methods. The physical $z = x + \mathrm{i}y$ and auxiliary ζ planes are shown in figure 8. The same notation is used as in section 2.1 where the general shock free cavitating problem is formulated. The velocity absolute value on the free surfaces is $v_0 = v_\infty\sqrt{1+\sigma}$. where $\sigma \geq 0$ is cavitation number. The region occupied by the fluid is bounded by solid segments $[AC]$, $CO]$ and $[OB]$ and cavity surfaces (BE) and (AE), point E is that at infinity (second Riemann sheet). Point at infinity F (first Riemann sheet) has the image $\zeta_0 = b + \mathrm{i}c$ and stagnation point F – image $\zeta_\infty = d + \mathrm{i}f$ on the first quadrant of the auxiliary ζ-plane.

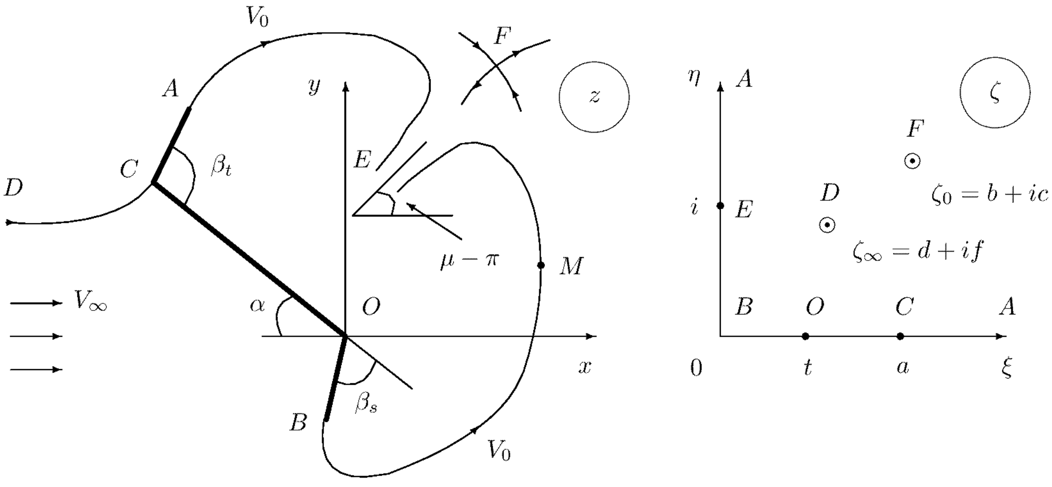


Figure 8: Flow pattern and auxiliary ζ-plane for the nonlinear shock free flat plate problem.

With the correspondence between the planes, the Chaplygin method allows us to write down the following exact solution:

$$\chi^{\mathrm{n}}(\zeta) = \frac{\mathrm{d}w}{v_0\mathrm{d}z} = e^{\mathrm{i}(\alpha-\gamma)}\left(\frac{\zeta-a}{\zeta+a}\right)^{\gamma/\pi}\left(\frac{\zeta-t}{\zeta+t}\right)^{\beta/\pi}\frac{(\zeta-\zeta_0)\,(\zeta-\bar{\zeta}_0)}{(\zeta+\zeta_0)\,(\zeta+\bar{\zeta}_0)}\,; \tag{28}$$

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = N\frac{\zeta(\zeta^2-a^2)(\zeta^2-\zeta_0^2)(\zeta^2-\bar{\zeta}_0^2)}{(\zeta^2+1)(\zeta^2-\zeta_\infty^2)^2(\zeta^2-\bar{\zeta}_\infty^2)^2}\,, \tag{29}$$

wherefrom

$$z(\zeta) = \frac{N}{V_0}\,e^{\mathrm{i}(\gamma-\alpha)}\int\limits_t^\zeta \frac{\zeta\,(\zeta+a)^{1+\gamma/\pi}\,(\zeta-a)^{1-\gamma/\pi}}{(\zeta^2+1)}\left(\frac{\zeta+t}{\zeta-t}\right)^{\beta/\pi}\cdot\frac{(\zeta+\zeta_0)^2(\zeta+\bar{\zeta}_0)^2}{(\zeta^2-\zeta_\infty^2)^2(\zeta^2-\bar{\zeta}_\infty^2)^2}\,d\zeta\,. \tag{30}$$

Five conditions

$$z_C = z(a) = l\, e^{\mathrm{i}(\pi-\alpha)}\,; \quad z_B = z(0) = \varepsilon\, e^{-\mathrm{i}(\alpha+\beta)}\,; \quad z_A - z_C = z(\infty) - z(a) = l_\mathrm{w}\, e^{\mathrm{i}(\gamma-\alpha)}\,;$$

$$\chi^\mathrm{n}(\zeta_\infty) = \frac{v_\infty}{v_0}\,; \qquad \oint\limits_{\zeta_\infty} \frac{\mathrm{d}z}{\mathrm{d}\zeta}\,\mathrm{d}\zeta = 0$$

generate a system of seven nonlinear equations in eight real unknowns a, t, $\zeta_0 = b + \mathrm{i}c$, $\zeta_\infty = d + \mathrm{i}f$, N and l_w. An additional eighth condition is to be used to close the problem, namely connected with direction of the re-entrant jet at point E. Assume that angle μ is given, then the condition is

$$\chi^\mathrm{n}(\mathrm{i}) = e^{-\mathrm{i}\mu}\,,$$

or

$$\theta(1) = \mu\,, \tag{31}$$

where $\theta(\eta)$ denotes direction of velocity vector

$$\theta(\eta) = -\arg\chi(\mathrm{i}\eta) = -\alpha + \gamma - \frac{2\gamma}{\pi}\arctan\frac{a}{\eta} - 2\arctan\frac{b}{\eta - c} - 2\arctan\frac{b}{\eta + c} - \frac{2\beta}{\pi}\arctan\frac{t}{\eta}\,. \tag{32}$$

The pressure distribution coefficient is

$$C_p^\mathrm{n}(\xi) = 1 - |\chi^\mathrm{n}(\xi)|^2 = 1 - \left|\frac{\xi - a}{\xi + a}\right|^{2\gamma/\pi} \left(\frac{\xi^2 - 2b\xi + b^2 + c^2}{\xi^2 + 2b\xi + b^2 + c^2}\right)^2 \left|\frac{\xi - t}{\xi + t}\right|^{2\beta/\pi}\,, \tag{33}$$

where $\xi = \mathrm{Re}\,(\zeta)$ is connected to z through relation (30). Lift and drag coefficients are

$$C_F = C_D + \mathrm{i}C_L = -\frac{\mathrm{i}}{l}\int\limits_0^\infty C_p^\mathrm{n}(\xi)\frac{\mathrm{d}z}{\mathrm{d}\xi}\,\mathrm{d}\xi = -\frac{\mathrm{i}}{l}(1+\sigma)\left(z_A - z_B - \frac{1}{v_0^2}\int\limits_0^\infty \frac{\mathrm{d}w}{\mathrm{d}\zeta}\overline{\frac{\mathrm{d}w}{\mathrm{d}z}}\,\mathrm{d}\xi\right)\,. \tag{34}$$

On the other hand, using residue theory, one arrives at the following relationships (which are correct for an arbitrary cavitating hydrofoil):

$$C_D = \frac{2q}{v_\infty l}\left(1 - \frac{v_0}{v_\infty}\cos\mu\right)\,; \qquad C_L = \frac{2q}{v_\infty l}\left(\frac{\Gamma}{q} - \frac{v_0}{v_\infty}\sin\mu\right)\,, \tag{35}$$

where $q = 2\delta_e v_0$ denotes the flow rate in the re-entrant jet E (δ_e is *a half* of the width of the jet) and Γ is circulation along a large contour completely surrounding the cavitating foil and the cavity and enclosing most of the flow. Note that

$$\mathrm{res} = \oint\limits_{u_\infty} \frac{\mathrm{d}F}{\mathrm{d}\zeta}\mathrm{d}\zeta = \Gamma + iq$$

and, moreover,

$$q = \pi N \frac{(1+a^2)\,\left((1+b^2-c^2)^2 + 4b^2c^2\right)}{2\left((1+d^2-f^2)^2 + 4d^2f^2\right)^2}\,.$$

Another form of the force coefficient $C_F = C_D + \mathrm{i}C_L$ is

$$C_F = \frac{4\pi}{v_\infty}\left(-\sqrt{1+\sigma}\,\mathrm{e}^{\mathrm{i}\mu}\mathrm{Re}\,(\mathrm{res}) + \overline{\mathrm{res}}\right)\,. \tag{36}$$

Let us analyse the solution to the problem under consideration in the case of $\alpha \to 0$, $\sigma \to 0$, $\varepsilon/l \to 0$ and $l_\mathrm{w}/l \to 0$, that is under assumption that hydrofoil brings small perturbations into the inflow. Thorough asymptotic analysis of the problem is given in [1] and below just final results are shown.
It is clear from what was done above that $a \to +\infty$, $\zeta_0 \to \mathrm{i}$ ($b \to 0$ and $c \to 1$) and $t \to 0$ if $\beta = \mathrm{O}(1)$ and $\gamma = \mathrm{O}(1)$.

Taking account of only linear terms in equations (28)–(32) under those circumstances results in $(c-1) \ll b$ (if $\mu = \pi + \mathrm{O}(\alpha)$) and

$$\sigma \sim 4fd_0\alpha + \frac{8\beta d_0}{\pi}\frac{1}{\sqrt{R}}\sqrt{\frac{\varepsilon}{l}}\,; \qquad \frac{L_n}{l} \sim 4d_0^2 f^2$$

$$\frac{l_{\mathrm{w}}}{l} \sim \frac{\pi^4}{4\gamma^4}\frac{Q}{(f^2+d_0^2)^2}\left(d_0\frac{\sigma}{2} + f\alpha + \frac{\beta}{\pi}\frac{1}{\sqrt{R}}\sqrt{\frac{\varepsilon}{l}}\right)^4 \tag{37}$$

$$\frac{\delta_e}{l} \sim \frac{\pi}{16}\frac{1}{f^2 d_0^2}\left(f\,\frac{\sigma}{2} - d_0\,\alpha - \frac{2\beta}{\pi}\,fd_0\,\frac{1}{\sqrt{R}}\sqrt{\frac{\varepsilon}{l}}\right)^2$$

$$\frac{\mathrm{d}w}{v_\infty \mathrm{d}\tilde{z}} \sim 1 - \frac{2\gamma}{\pi}\frac{\zeta}{a} - 4b\,\frac{\zeta}{\zeta^2+1} - \frac{2\beta}{\pi}\frac{t}{\zeta}; \qquad \frac{\mathrm{d}w}{\mathrm{d}\zeta} \sim \frac{-Na^2\zeta(\zeta^2+1)}{\left(\zeta^4 + 2\zeta^2 + (f^2+d_0^2)^2\right)^2}, \tag{38}$$

where L_n denotes 'nonlinear' cavity extent, $\tilde{z} = \mathrm{e}^{i\alpha}z$ and

$$d_0 = \sqrt{f^2-1}\,; \qquad R = \int_0^1 \xi\left(\frac{1+\xi}{1-\xi}\right)^{\beta/\pi}\mathrm{d}\xi\,; \qquad Q = \int_0^1 \xi(1-\xi^2)\left(\frac{1+\xi}{1-\xi}\right)^{\gamma/\pi}\mathrm{d}\xi\,. \tag{39}$$

The conformal mapping of the first quadrant of ζ–plane onto the upper semi–plane u has the following form

$$\zeta = \mathrm{i}\sqrt{\frac{u+u_0}{u}}\;,$$

where $u_0 = 2d_0 f \sim \sqrt{L_n/l}\,$. Substituting this expression into (38) gives

$$\frac{\mathrm{d}w}{v_\infty \mathrm{d}\tilde{z}} \sim 1 - \mathrm{i}\,\frac{2\beta}{\pi}\frac{tu_0}{\sqrt{u\,(u+u_0)}} + \mathrm{i}\sqrt{\frac{u+u_0}{u}}\left(\frac{4b}{u_0}\,u - \frac{2\gamma}{\pi}\frac{1}{a} + \frac{2\beta}{\pi}\,t\right) \tag{40}$$

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}u} \sim (L_n + l)\,\frac{2u}{(u^2+1)^2}\,. \tag{41}$$

Bearing in mind that 'linear' L and 'nonlinear' L_n cavity extent are connected to each other through the relation (due to a shift of coordinate systems)

$$L = 1 + \frac{L_n}{l}$$

and $L = \cos^{-2}(\tau/2)$, we find that

$$d_0 \to \frac{\sin\dfrac{\pi-\tau}{4}}{\sqrt{\sin\dfrac{\tau}{2}}}\,; \qquad f \to \frac{\cos\dfrac{\pi-\tau}{4}}{\sqrt{\sin\dfrac{\tau}{2}}}\,.$$

Substituting these limits into asymptotic expansions obtained above gives expressions coinciding with those reduced from outer (2) and composite (27) solutions in the case of a supercavitating shock free flat plate when $f(x) = -\alpha x$ and $\mathcal{D}_k = \mathcal{E}_k = 0$. Moreover, the asymptotic and nonlinear approaches give the similar formulae for the width of re-entrant jet $2\delta_e$ and upper wetted portion of the leading edge l_{w}/l (compare (26) and (25) with (37)).

2.6. Numerical results

Mathematica for Windows computer mathematical environment was used to obtain all the numerical results shown in this section.

Flow pattern and hydrodynamic coefficients are given in figure 9 for nonlinear theory (NLT) and matched asymptotics (MAE) for a shock free flat plate. The flow parameters are chosen to be $\alpha = 10^{\mathrm{o}}$, $\sigma = 0.5$, $\bar{\varepsilon} = 0.05$, $\beta = 90^{\mathrm{o}}$, $\gamma = 60^{\mathrm{o}}$ and $\mu = 200^{\mathrm{o}}$. The position of stagnation point F is also shown.

Pressure distribution coefficient C_p is depicted for such a flat plate in figure 10 (for the same set of the flow parameters). Numerical results for the linear theory (outer asymptotic expansion $\chi^{\mathrm{o}}(z)$) is shown as well.